• **Skyrme Model**: Solitons. The SU(2) Skyrme model. The “hedgehog skyrmion” and its quantization. Extension to $N_f > 2$ flavors. Multibaryon configurations.
Solitary waves and solitons

Let’s consider the most simple wave equation: scalar field in 1+1 dim

\[ \partial^\mu \partial_\mu \phi(x,t) = \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi(x,t) = 0 \]

We take \( c = 1 \)

The solution of this equation satisfy that

i) Any well-behaved function of the form \( f(x \pm t) \) is a solution. In particular, if we choose a localized function \( f \) we can construct a localized wave packet that will travel with uniform velocity \( \pm 1 \), and no distortion in shape: if we Fourier decompose it, each component travels with the same velocity \( \omega/k = 1 \)

ii) Since the equation is linear, given two localized wave-packet solutions \( f_1(x - t) \) and \( f_2(x + t) \), the sum \( f_3(x, t) = f_1(x - t) + f_2(x + t) \) is also a solution. At large negative time (\( t \to -\infty \)) \( f_3(x, t) \) consists of the two packets widely separated and approaching each other essentially undistorted. At finite \( t \), they collide. But after collision they will asymptotically (as \( t \to +\infty \)) separate into the same two packets retaining their original shapes and velocities.

The validity of these properties is due to the fact that the equation is linear and non dispersive.
Note that the addition of even the simplest kinds of terms to this equation tends to destroy these nice features, even in (1 + 1) dimensions. For example, if we include a mass term

\[
\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\phi(x,t) = 0
\]

The equation is still linear. However, it is dispersive \( \omega^2(k) = k^2 + m^2 \). Thus, each Fourier component of a wave packet will travel with a different velocity \( \omega(k)/k \).

The question is: Is it possible to find some equations where both dispersive and non-linear terms are present, and where their effects might balance each other in such a way that some special solutions do essentially enjoy features (i) and (ii)?

The answer is: Yes, this could happen in 1 or more spatial dimensions.

The solutions are generally called “solitary waves” (if they also satisfy (i)) or “solitons” (if in addition they satisfy (ii)), although these definitions are not universally accepted.
In general it is convenient to work in terms of localized energy densities instead of localized field distributions.

We will define as a “solitary wave” a non-singular solution of a non-lineal field equation whose associated energy density is localized and has a space-time dependence of the form

\[ \varepsilon(\vec{x}, t) = \varepsilon(\vec{x} - \vec{u} t) \]

\( \vec{u} \) velocity vector

We must note that any localized time independent solution is automatically a solitary wave with \( \vec{u} = 0 \). Given Lorentz (or Galilean) invariance once one knows an static solution, time dependent solution are easily obtained by “boosting”

In general, to prove that a solitary wave is a soliton is a complicated task, and thus we will not pay too much attention to that in what follows.
Example: Solitary waves in 1+1 dim, the “kink”

Let us consider

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \]

where \( V(\phi) \) has several degenerate minima. The associated conserved energy is

\[
E(\phi) = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + V(\phi) \right]
\]

It is clear that the energy is minimum (zero) if

\[ \phi(x, t) = \text{constant} = \phi_i \]

where \( \phi_i \) is one of the minima of the potential. The associated Euler-Lagrange equation is

\[ \ddot{\phi} - \phi'' = -\frac{\partial V}{\partial \phi} \]
As seen, to look for solitary waves it is enough to consider static solutions. In that case

\[
E(\phi) = \int_{-\infty}^{+\infty} dx \left[ \frac{1}{2} \phi'^2 + V(\phi) \right]
\]

This implies that any solution with finite energy and localized energy density must approach, as \( x \to \pm \infty \), one of the minima \( \phi_i \). Otherwise \( E \) would be divergent. Thus, we must find solutions that satisfy

\[
\phi'' = \frac{\partial V}{\partial \phi} \quad \text{with} \quad \lim_{x \to \pm \infty} \phi(x) = \phi_i
\]

This equation can be explicitly integrated. Multiplying by \( \phi' \),

\[
\phi' \phi'' = \frac{\partial V}{\partial \phi} \phi' \quad \rightarrow \quad \frac{1}{2} \frac{\partial \phi'^2}{\partial x} = \frac{\partial V}{\partial x} \quad \rightarrow \quad \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\partial \phi'^2}{\partial x} = \int_{-\infty}^{\infty} dx \frac{\partial V}{\partial x} \quad \rightarrow \quad \frac{1}{2} \phi'^2 = V \quad \rightarrow \quad \frac{d \phi}{dx} = \pm \sqrt{2V}
\]

Where in the previous to the last term we have used \( \phi'(-\infty) = V(\phi(-\infty)) = 0 \). Finally, performing a new integration with respect to \( x \) we get

\[
x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d \phi}{\sqrt{2V(\phi)}}
\]
Let us consider now the particular case

\[ V(\phi) = \frac{\lambda}{4} \left( \phi^2 - m^2 / \lambda \right)^2 \]

which has two degenerate minima \( \phi_i = \pm \nu \) with \( \nu = (m/\lambda^{1/2}) \). The associated EL eq. is

\[ \phi'' + m^2 \phi - \lambda \phi^3 = 0 \]

non-linear and dispersive

whose solution is

\[ x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{\lambda / 2} \left( \phi^2 - m^2 / \lambda \right)} \]

Taking \( \phi(x_0) = 0 \)

\[ \phi(x) = \pm \frac{m}{\sqrt{\lambda}} \tanh \left[ \frac{m}{\sqrt{2}} (x - x_0) \right] \]

+ kink
- anti-kink
The energy density is \[ \varepsilon(x) = \frac{1}{2} \phi'^2 + V(\phi) = 2 V(\phi) \], from where we get

\[ \varepsilon(x) = \frac{m^4}{2\lambda} \left( \text{Sech} \left[ \frac{m}{\sqrt{2}} (x - x_0) \right] \right)^4 \]

The total static energy is

\[ E_0 = \int_{-\infty}^{\infty} dx \varepsilon(x) = \frac{2\sqrt{2}}{3} \frac{m^3}{\lambda} \]

\[ E_0 \text{ goes as the inverse of the } \text{“coupling constant” } \lambda \text{ (non-perturbative)} \]

This solution has the characteristics of a particle. If we propose a time dependent solution of the form

\[ \phi(x,t) = \phi(x - X_0(t)) \]

we have

\[ \dot{\phi} = -\dot{X}_0 \phi' \quad \rightarrow \quad \Delta E = \int_{-\infty}^{\infty} dx \frac{1}{2} \phi'^2 = \dot{X}_0^2 \int_{-\infty}^{\infty} dx \frac{1}{2} \phi''^2 = \frac{1}{2} E_0 \dot{X}_0^2 \]

from where

\[ E = E_0 + \Delta E = E_0 + \frac{1}{2} E_0 X_0^2 \]

\[ E_0 \text{ mass of the particle} \]
Finally, we must stress that there is a conserved current associated with this type of solutions. In the present case

\[ j^\mu = \varepsilon^{\mu\nu} \partial_\nu \phi \rightarrow \partial_\mu j^\mu = 0 \]

We must note that the conservation of \( j^\mu \) is independent of the equations of motion → *topological current*

The associated charge is

\[ Q = \int_{-\infty}^{\infty} dx \ j^0 = \int_{-\infty}^{\infty} dx \ \frac{\partial \phi}{\partial x} = \phi(+\infty) - \phi(-\infty) = \begin{cases} 2m / \sqrt{\lambda} & \text{kink} \\ 0 & \text{trivial solution} \\ -2m / \sqrt{\lambda} & \text{antikink} \end{cases} \]

This charge distinguishes between topological classes: *it is not possible to continuously deform a solution with a given value of \( Q \) into another one with a different value of charge since this costs infinite energy.*
The SU(2) Skyrme model

What we have discussed about large $N_c$ QCD led Witten to suggest, at the end of the 70’s, that it is possible to consider baryons as solitons of a non-linear meson theory. Indeed, a model of this type has been already proposed by T. Skyrme at the beginning of the 60’s.

A convenient starting point is the SU(2)$_f$ non-lineal $\sigma$-model

$$\mathcal{L}_{nl\sigma} = \frac{f_\pi^2}{4} Tr \left[ \partial_\mu U \partial^\mu U^\dagger \right]$$

where

$$U = \exp \left[ i \frac{\vec{\tau} \cdot \vec{\pi}}{f_\pi} \right]$$

If pions are considered as small amplitude fluctuations around the trivial vacuum $U=1$, the exponential can be expanded and one obtains a theory for weakly interacting pions (leading term of ChPT)

$$\mathcal{L}_{nl\sigma} = \frac{1}{2} \partial_\mu \pi \partial^\mu \pi + \frac{1}{6 f_\pi^2} \left[ (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 - \vec{\pi}^2 \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} \right] + O(\pi^6)$$
For general $U$-field configurations the non-lineal $\sigma$-model hamiltonian is

$$\mathcal{H} = \frac{f_\pi^2}{4} \text{Tr} \left[ (\dot{U})^2 + (\vec{\nabla}U)^2 \right]$$

Thus, the associated energy is given by

$$E = \int d^3x \mathcal{H} = \frac{f_\pi^2}{4} \int d^3x \text{Tr} \left[ (\partial_0 U)^2 + (\vec{\nabla}U)^2 \right]$$

As in the case of the kink, to look for soliton solutions we will consider static solutions, in which case $\partial_0 U=0$. Therefore to have finite energy solution it must happen that

$$\lim_{|x| \to \infty} \vec{\nabla}U(\vec{x}) = 0$$

which implies

$$\lim_{|x| \to \infty} U(\vec{x}) = \text{constant} = U_0$$

Generally, one takes $U_0=1$ which is equivalent to $\sigma_0(\infty)=1$, $\pi_0^a(\infty)=0$
We notice now that a 3 dim space in which all the points at infinity are identified with the same point is topologically equivalent to an 4 dim sphere of infinite radius $S^3_\infty$ (i.e. $\mathbb{R}^3$ is compactified into $S^3_\infty$).

On the other hand, note that using the properties of Pauli matrices $[\tau_i, \tau_j] = i \varepsilon_{ijk} \tau_k$ ; $\{\tau_i, \tau_j\} = \delta_{ij}$ we can write

$$U = c_0 + i \bar{\tau} \cdot \bar{c} \quad \text{where} \quad c_0 = \cos \left( \frac{\pi}{f_\pi} \right) \quad ; \quad \bar{c} = \hat{\tau} \sin \left( \frac{\pi}{f_\pi} \right)$$

Since $U$ is a unitary matrix (i.e. $UU^\dagger = 1$) this implies

$$c_0^2 + \bar{c}^2 = 1 \quad \rightarrow \quad U \quad \text{takes values on a 3-sphere in 4 dim}$$

Therefore, $U(\vec{x})$ with the condition $U(\infty) = 1$ can be understood as a map $S^3 \rightarrow S^3$. These maps can be classified according a topological number called “winding number” $W$. Maps with different $W$ belong to different “homotopy classes”.
To understand better this point, let us consider in the simpler case of $S^1 \to S^1$. Then, we have

Trivial map $U(x) = 1$

Map that can be continuously deformed into the trivial map

Map that covers the entire target space. It can **NOT** be deformed into the trivial map
The "winding number" measures how many times one covers the target space when one covers once the domain. This is a conserved number due to topological reasons. As in the case of the kink, here it is also possible to define a topological current. In the present case

\[
W_\mu = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\alpha\beta} Tr \left[ U^\dagger \partial^\nu U U^\dagger \partial^\alpha U U^\dagger \partial^\beta U \right]
\]

The associated charge \( W = \int d^3 x W_0 \) is precisely the "winding number".

Therefore we see that, in principle, the non-linear \( \sigma \)-model supports non-trivial fields configurations.
The following question is whether these non-trivial configurations are energetically stable and localized. Let us suppose that we find a non-trivial solution whose energy is

\[
E = \frac{f_\pi^2}{4} \int d^3 x \ Tr \left[ \left( \nabla U(x) \right)^2 \right]
\]

If we now perform a re-scaling of the coordinates \( U(x) \rightarrow U(x/\lambda) \) we have

\[
E(\lambda) = \lambda \frac{f_\pi^2}{4} \int d^3 x' \ Tr \left[ \left( \nabla U(x') \right)^2 \right] \quad \text{where} \quad x' = x/\lambda
\]

Therefore, the energy minimum is obtained for \( \lambda \rightarrow 0 \), i.e. the size of the configuration shrinks to zero. The solution of this problem was suggested by T. Skyrme who added a quadratic term in derivatives to stabilize the soliton configurations

\[
\mathcal{L}_{Sk} = \frac{f_\pi^2}{4} \ Tr \left[ \partial_\mu U \partial^\mu U^\dagger \right] + \frac{1}{32e^2} \ Tr \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2
\]

It is easy to see that the contribution from the quadratic term scales as \( 1/\lambda \) what leads to stable localized solutions.
Skyrme’s idea was to identify the “winding number” with the baryonic number. Therefore, to find solutions associated to a baryon (e.g. a nucleon) we have to find solutions that minimize the static energy and have $W=1$. In order to do that Skyrme proposed the “hedgehog” ansatz (usually know as “Skyrmion”)

$$U_{H}(\vec{r}) = \exp\left( i \, F(r) \, \vec{\tau} \cdot \vec{r} \right)$$

$F(r)$ is the skyrmion profile which should be obtained from the corresponding E-L eq.

Replacing in the equation for $W_0$ one gets

$$W_0 = -\frac{1}{2\pi^2} \frac{\sin^2 F}{r^2} F'$$

that implies

$$W = \int d^3x \, W_0 = -\frac{2}{\pi} \int_{F(0)}^{F(\infty)} dF \sin^2 F = \frac{2}{\pi} \left[ \frac{F(0)}{2} - \frac{\sin 2F(0)}{4} \right]$$

where in the last step we have used $F(\infty)=0$. Therefore, if $F(0)= n \pi$ we get

$$W = B = n$$
Therefore, for $B=1$ we have to find the solution of the E-L eq. which results from minimizing the corresponding soliton mass with the boundary conditions $F(0)=\pi$, $F(\infty)=0$. The explicit form of the skyrmion mass is

\[
M_H = 4\pi f_\pi^2 \int_0^\infty dr \left\{ r^2 F''^2 + 2\sin^2 F + \frac{1}{2\left( e f_\pi \right)^2} \sin^2 F \left( 2 F'^2 + \frac{\sin^2 F}{r^2} \right) \right\} = O(N_c)
\]

Defining $\tilde{r} = e f_\pi r$, the resulting E-L equation is

\[
F''\left( \tilde{r}^2 + 2\sin^2 F \right) + 2F'\tilde{r} + (F'^2 - 1)\sin 2F - \frac{\sin^2 F \sin 2F}{\tilde{r}^2} = 0
\]

Solving this equation numerically one obtains the skyrmion profile as a function of the dimensionless variable $\tilde{r}$.
\( U_H \) corresponds to a static solution which does not have good quantum numbers of spin and isospin, only their sum \( \vec{K} = \vec{L} + \vec{I} \) (so-called “grand spin”) is good quantum number. This is due to the fact that in the skyrmion the spin and isospin spaces are coupled.

In the large \( N_c \) language the skyrmion is in the “plane wave” basis. To recover the good spin and isospin quantum numbers we have to project to the “spherical basis”, i.e. we have to “rotate” the skyrmion. This is equivalent to rotate a deformed nucleus to recover good angular momentum (“cranking”).

To perform this operation we introduce collective coordinates in isospin space

\[
U(\vec{r},t) = A(t) U_H(\vec{r}) A^\dagger(t)
\]

\( A(t) \) is an SU(2) matrix

\[
A(t) = a_0(t) + i \vec{a}(t) \cdot \vec{\tau}
\]

with \( a_0^2(t) + \vec{a}(t) \cdot \vec{a}(t) = 1 \) can be conveniently parametrized as

\[
A(t) = \exp \left( i \Phi(t) \frac{\tau_3}{2} \right) \exp \left( i \Theta(t) \frac{\tau_2}{2} \right) \exp \left( i \Psi(t) \frac{\tau_3}{2} \right)
\]
Replacing in the Skyrme lagrangian one gets an additional rotational contribution

\[ L_{rot} = \frac{\zeta}{2} \, \vec{\omega} \cdot \vec{\omega} \]

\[ \vec{\omega} = -i \text{Tr} \left[ A^\dagger (t) \, \dot{A}(t) \, \vec{r} \right] \]

\[ \zeta = \frac{8 \pi f_\pi^2}{3} \int_0^\infty dr \, r^2 \sin^2 F \left( 1 + \frac{1}{e^2 f_\pi^2} \left( F^2 + \frac{\sin^2 F}{r^2} \right) \right) = O(N_c) \]

Written in terms of the Euler angles \((\Phi, \Theta, \Psi)\) this reads

\[ L_{rot} = \frac{\zeta}{2} \left( \dot{\Phi}^2 + \dot{\Theta}^2 + 2 \cos \Theta \, \dot{\Phi} \, \dot{\Psi} \right) \]

This kinetic term has the general form \( \dot{\zeta}_i \, g_{ij}(\zeta) \, \dot{\zeta}_j \) where \( \zeta_i \) are the dynamical variables and \( g_{ij}(\zeta) \) is the coordinate dependent metric. Generalizing the usual quantization rules (i.e. \( p_x \rightarrow -i \partial / \partial x \)) one obtains that its contribution to the quantum Hamiltonian is

\[ \frac{1}{2} \, \dot{\zeta}_i \, g_{ij}(\zeta) \, \dot{\zeta}_j \rightarrow -\frac{1}{\sqrt{g}} \frac{\partial}{\partial \zeta_i} (g^{-1})_{ij} \sqrt{g} \frac{\partial}{\partial \zeta_j} \]

where \( g = \det(g_{ij}) \)
In this particular case this procedure leads to

\[
H_{\text{rot}} = -\frac{1}{2\Im} \left[ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \sin \Theta \frac{\partial}{\partial \Theta} + \frac{1}{\sin^2 \Theta} \left( \frac{\partial^2}{\partial \Phi^2} - 2 \cos \Theta \frac{\partial^2}{\partial \Phi \partial \Psi} + \frac{\partial^2}{\partial \Psi^2} \right) \right]
\]

However, this differential operator has exactly the form of \( J^2 \) when written in terms of the Euler angles. As well known the corresponding eigenfunctions are the Wigner D-functions. Namely,

\[
H_{\text{rot}} D^{j}_{m,m'}(\Phi, \Theta, \Psi) = -\frac{j(j+1)}{2\Im} D^{j}_{m,m'}(\Phi, \Theta, \Psi)
\]

To identify the quantum numbers \( j, m, m' \) one can construct the Noether currents that arise from the infinitesimal transformations

\[
U \rightarrow U - \tilde{\alpha} \cdot \left[ U, \frac{\tau}{2} \right] + \ldots \quad \text{(isospin)}
\]
\[
U \rightarrow U - \tilde{\beta} \cdot \left[ U, -i \vec{r} \times \vec{\nabla} \right] + \ldots \quad \text{(spin)}
\]

In this way one gets

\[
I = J = j \quad I_3 = m \quad J_3 = -m'
\]
The properly normalized eigenstates are then

\[ |J = I, I_3, J_3 > = \left( \frac{2J + 1}{8\pi^2} \right)^{1/2} D_J^{I_3, -J_3}(\Phi, \Theta, \Psi) \]

The energy spectrum has the form

\[ M(J = I) = M_{sol} + \frac{J(J + 1)}{2\sqrt{3}} \]

Identical to the one resulting from the large \( N_c \) expression

\[ \hat{M} = m_1 N_c \hat{1} + \frac{m_{-1}}{N_c} \hat{J}^2 + \mathcal{O}\left(1 / N_c^2\right) \]

But now we have predictions for the constants \( m_1 \) and \( m_{-1} \).
The model has only two parameters $f_\pi$ and $e$. If we take them as free parameters and adjust them to reproduce the empirical values of $M_N$ and $M_\Delta$ it results:

$$f_\pi = 54 \text{ MeV}, e = 4.84$$

To be more realistic a pion mass term is added

$$\mathcal{L}_{sb} = \frac{f_\pi^2 m_\pi^2}{4} Tr[U + U^\dagger - 2]$$

Agreement within 30 %, except for $g_A$ which is a factor 2 too small.

If one takes $f_\pi = 93 \text{ MeV}$ and $e$ is determined from $\pi\pi$ scattering, the skyrmion mass turns out to be too large ($\sim 1.5 \text{ GeV}$). However, when one takes into account pion fluctuations the contribution from Casimir energy tends to solve this problem. The quality of the rest of the predictions is quite similar.

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Extension to flavor SU(3)

To extend the Skyrme model to SU(3) one has to add additional terms to the Skyrme lagrangian

$$\mathcal{L}_{sk} = \frac{f_\pi^2}{4} Tr \left[ \partial_\mu U \partial^\mu U^\dagger \right] + \frac{1}{32 e^2} Tr \left[ U^\dagger \partial_\mu U , U^\dagger \partial_\nu U \right]^2 U \in SU(3)_f$$

Firstly, one has to take care of the flavor breaking due to the difference between the pion and kaon mass. The simplest way to implement this is to add a term

$$\mathcal{L}_{sb} = \frac{f_\pi^2}{12} (m_\pi^2 + 2m_K^2) Tr[U + U^\dagger - 2] + \frac{f_\pi^2}{2\sqrt{3}} (m_\pi^2 - m_K^2) Tr[\lambda_8 (U + U^\dagger)]$$

Secondly, one has to include the Wess-Zumino-Witten (WZW) term

$$\Gamma_{WZ} = -i \frac{N_c}{240\pi^2} \int d^5 x \, \varepsilon^{\mu\nu\alpha\beta\gamma} \, Tr \left[ U^\dagger (\partial_\mu U) U^\dagger (\partial_\nu U) U^\dagger (\partial_\alpha U) U^\dagger (\partial_\beta U) U^\dagger (\partial_\gamma U) \right]$$

This terms is responsible for eliminating spurious symmetry of nl σ-model (invariance $U \rightarrow U^\dagger$) that would forbid some QCD allowed decays such as $K^+ K^- \rightarrow 3 \pi$
SU(3) collective quantization approach

In this approach one performs a trivial SU(3) embedding of the SU(2) skyrmion and consider SU(3) rotations in flavor space

\[
U(x,t) = A(t) \begin{pmatrix}
\exp \left( F(r) \vec{\tau} \cdot \vec{r} \right) & 0 \\
0 & 1
\end{pmatrix} A^\dagger(t) \quad A \in SU(3)_f
\]

Of the 8 collective coordinates, one (the associated with rotations around the 8\(^{\text{th}}\) axis) leave the solution invariant. This will lead to a constraint. Replacing into the lagrangian

\[
L = -M_{\text{sol}} + \frac{\tilde{\zeta}_\pi}{2} \sum_{i=1,2,3} \omega_i^2 + \frac{\tilde{\zeta}_K}{2} \sum_{k=4,\ldots,7} \omega_k^2 - \frac{N_c}{2\sqrt{3}} \omega_8 - \frac{\gamma_{sb}}{2} \left( 1 - D_{88} \right)
\]

where the “pionic” and kaonic” moment of inertias \(\tilde{\zeta}_\pi\) and \(\tilde{\zeta}_K\), respectively, and the symmetry breaking parameter \(\gamma_{sb}\) are explicit functions of the soliton profile \(F(r)\). In addition

\[
A^\dagger \dot{A} = i \sum_{a=1}^{8} \lambda^a \omega^a \quad \text{and} \quad D_{88} = \frac{1}{2} Tr \left[ \lambda^8 A \lambda^8 A^\dagger \right]
\]
Defining the conjugate momenta in the usual $F^a = \partial L / \partial \omega^a$ we obtain after a Lagrange transformation

$$H = M_{sol} + \frac{1}{2} \left( \frac{1}{3} - \frac{1}{2} \right) J^2 + \frac{1}{2} \sum_{a=1}^{8} F^a F^a - \frac{N_c^2}{24 \sqrt{3}} \left[ 1 - D_{88} \right]$$

The eigenfunctions of $H_{sym}^{coll}$ are the SU(3) D-functions $D_{\gamma_{I I3 ; \gamma R R I3 R}}^{(p,q)} (\alpha_1, ..., \alpha_8)$

The WZW term gives the constraint $F_8 = 2\sqrt{3} Y^R = N_c / 2\sqrt{3}$ which implies $Y^R = 1$. This, in turns, implies that the allowed values of $(p,q)$ should satisfy (for arbitrary baryon number $B$)

$$p + 2 q \frac{3}{3} = N_c \frac{3}{3} B + t$$

where $t$ is an integer. The lowest allowed states ($t = 0$) have $(p,q) = (1,1), (3,0)$, namely they are the 8 and 10

One can also see that $I^R$ (and therefore $J$) should be half integer (quantized solitons are fermions !)
To understand this point in detail let us consider $B=1$ and typical higher dimensional SU(3) multiplet (in this case $(p,q) = (5,2)$ and $\text{dim} = 81$).

State with maximum hypercharge $Y^{\text{(max)}}$ and, for that $Y$, the maximum $I_3^R$

SU(2) multiplet that satisfies the constraint $Y^{(R)} = \frac{N_c}{3} = 1$

We see that $Y^{\text{(max)}} = \frac{(p+2q)}{3}$ and that the $Y^R$ for any other SU(2) submultiplet differs by an integer $t \leq p + q$. Since an allowed multiplet should contain an SU(2) submultiplet with $Y^R = \frac{N_c}{3}$ we must have

$$\frac{(p+2q)}{3} = \frac{N_c}{3} + t$$

Given that $I_3^{R(\text{max})} = \frac{p}{2}$ for $Y^{(\text{max})}$ (i.e. $t=0$) we can determine $I_3^{R(\text{max})}$ for SU(2) multiplets corresponding to other values of $t$. In particular, for the one satisfying the above constraint we have

$$I_3^{R(\text{max})} = \frac{N_c}{2} + k$$

where $k = 2t - q$ for $t \leq q$ or $k = t$ otherwise

Thus, $I_3^R$ (which should be identified with $-J_3$) is half integer if $N_c$ is odd (for $B=1$).
So far we have neglected the symmetry breaking term $H_{sb}^{coll}$. One might be tempted to treat it as a perturbation. In such a case the collective contribution to baryon mass will be given by the eigenvalues of $H_{sym}^{coll}$ plus the diagonal matrix element of $H_{sb}^{coll}$. This happens to be a pretty bad approximation. $H_{sb}^{coll}$ should be diagonalized exactly (Yabu-Ando approach \textit{Nucl. Phys. B301}(1988)601). For that we put

$$\Psi_{\tilde{Y}I\tilde{I}z,\tilde{J}J_z} = \sum_{(p,q)} \beta_{(p,q)} D_{\tilde{Y}I,I_z,(1,\tilde{J},J_z)}^{(p,q)}$$

and find $\beta_{(p,q)}$ such that

$$\left[ \frac{1}{2\zeta_K} \sum_{a=1}^{8} F_a F^a + \frac{\gamma_{sb}}{2} (1 - D_{88}) \right] \Psi_{\tilde{Y}I\tilde{I}z,\tilde{J}J_z} = \epsilon_{\tilde{Y}I\tilde{I}z,\tilde{J}J_z} \Psi_{\tilde{Y}I\tilde{I}z,\tilde{J}J_z}$$

Then

$$M_{\tilde{Y}I\tilde{I}z,\tilde{J}J_z} = M_{sol} + \frac{1}{2} \left( \frac{1}{\zeta_{\pi}} - \frac{1}{\zeta_K} \right) J(J+1) - \frac{3}{8\zeta_K} + \epsilon_{\tilde{Y}I\tilde{I}z,\tilde{J}J_z}$$
For that we put

$$\Psi_{YIIz,1JJz} = \sum_{(p,q)} \beta_{(p,q)} D^{(p,q)}_{(Y,1z),(1,Jz)}$$

and find $\beta_{(p,q)}$ such that

$$\left[ \frac{1}{2\Theta_K} \sum_{a=1}^{8} F^a F^a + \frac{\gamma_{sb}}{2} (1 - D_{88}) \right] \Psi_{YIIz,1JJz} = \varepsilon_{YIIz,1JJz} \Psi_{YIIz,1JJz}$$

Then

$$M_{YIIz,1JJz} = M_{sol} + \frac{1}{2} \left( \frac{1}{\Theta_\pi} - \frac{1}{\Theta_K} \right) J(J+1) - \frac{3}{8\Theta_K} + \varepsilon_{YIIz,1JJz}$$

Typical result for the spectrum is (splittings with respect to Nucleon mass in MeV are given)

<table>
<thead>
<tr>
<th>Baryon</th>
<th>CCSM</th>
<th>expt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$</td>
<td>163</td>
<td>177</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>264</td>
<td>254</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>388</td>
<td>379</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>268</td>
<td>293</td>
</tr>
<tr>
<td>$\Sigma^*$</td>
<td>406</td>
<td>446</td>
</tr>
<tr>
<td>$\Xi^*$</td>
<td>545</td>
<td>591</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>680</td>
<td>733</td>
</tr>
</tbody>
</table>

Reasonable agreement.

Other SU(3) baryon properties (magnetic moments, etc) are also reasonably well described.
Not any flavor irrep is allowed. The lowest allowed are $8, 10, 10, \ldots$.

The first two (8 and 10) corresponds to the usual octet $(N, \Lambda, \Sigma, \Xi)$ and decuplet $(\Delta, \Sigma^*, \Xi^*, \Omega)$ baryons while the antidecuplet to the first set of exotics.

Although the existence of this multiplet was known since the 80’s in their 1997 article Diakonov et al. made a definite prediction about the mass of the $\theta^+$ and its width. They predicted 1530 MeV and a narrow width $\sim 30$ MeV. They assumed the $N$ of this multiplet to be a nucleon resonance which does not fit in the NRQM. Since has $S=+1$ it has to be a pentaquark !!

The possible experimental identification of $\theta^+$ in 2004 (Nakano et al) triggered a lot of interest on pentaquarks. Nowadays, this interest has faded out together with (empirical) evidence.
An alternative approach to treat strange baryons was suggested by Callan and Klebanov (Nucl. Phys. B262 (1985) 365) At the classical level they proposed to use the ansatz

\[ U = \sqrt{U_0} U_K \sqrt{U_0} \]

\[ U_0 = \exp(i F(r) \tau \cdot \hat{r}) \]

\[ U_K = \exp \left[ i \frac{\sqrt{2}}{f_K} \begin{pmatrix} 0 & K \\ K^\dagger & 0 \end{pmatrix} \right] \]

Here \( K \) is the usual kaon doublet. We replace this ansatz in the effective lagrangian assuming that due to “large” kaon mass only small amplitude fluctuations are allowed in the strange direction. Keeping up to quadratic term in \( K \) we get

\[ L = -M_{sol} + L_k \]

where

\[ L_K = \int d^3 r \left[ f \dot{K}^\dagger \dot{K} + i \lambda \left( \dot{K}^\dagger K - K^\dagger \dot{K} \right) + K^\dagger V K \right] \]

Here \( f, \lambda \) are functions of the radial coordinates and the soliton profile. In particular \( \lambda \) is originated by the WZW term. \( V \) is an operator which does not have time derivatives.
Because of the particular spin-isospin structure of the soliton background, it is convenient to partial-wave-decompose the kaon field $K$ and its momentum canonically conjugate ($\Pi = \partial L_K / \partial K$) in the basis ($\Lambda, l$). Here, $\Lambda$ is the sum of the kaon isospin and its orbital angular momentum. Integrating out angular variables, one gets for the hamiltonian density for each partial wave

$$\mathcal{H}_\Lambda = \frac{1}{f} \Pi^\dagger_\Lambda \Pi_\Lambda + h K^\dagger_\Lambda K'_\Lambda + \left( m_K^2 + V^\text{eff}_\Lambda + \frac{\lambda^2}{f} \right) K^\dagger_\Lambda K_\Lambda + i \frac{\lambda}{f} \left[ K^\dagger_\Lambda \Pi_\Lambda - \Pi^\dagger_\Lambda K_\Lambda \right]$$

Here, $h$ and $V^\text{eff}_\Lambda$ are radial functions. This hamiltonian density is quadratic in the dynamical variables; therefore it can be diagonalized exactly. The corresponding eigenvalue equation is

$$\left[ -\frac{1}{r^2} \partial_r \left( r^2 h \partial_r \right) + m_K^2 + V^\text{eff}_\Lambda - f \epsilon^2_{\Lambda s} (n) + 2 s \lambda \epsilon_{\Lambda s} (n) \right] k_{\Lambda s} (r, n) = 0$$

$s = \pm 1$ is the kaon (or antikaon) strangeness and $n$ is the radial quantum number. As it turns out, for reasonable parameters of the model the lowest eingenergy corresponds to $s = -1$, $n=1$ and $(\Lambda, l) = (1/2, 1)$. In what follows we only consider that state and drop all the indexes.
To describe the splittings between baryons with the same strangeness but different
spin and/or isospin, we have to go beyond the static ansatz. We need to take into
account rotational zero-modes of the soliton. This can be done by introducing the
SU(2) collective coordinates $A(t)$

\[
\begin{align*}
N(\vec{r},t) &= A(t) N(\vec{r}) A^\dagger(t) \\
K(\vec{r},t) &= A(t) \tilde{K}(\vec{r},t)
\end{align*}
\]

where $\tilde{K}$ kaon in soliton frame

Inserting in the effective action we get the following additional $\mathcal{O}(1/N_c)$ contribution

\[
2 \bar{\omega} \cdot \left( \mathcal{S} \bar{\omega} - c \bar{J}_K \right)
\]

where $\bar{J}_K$ is the spin of the bound kaon and $c$ is the hyperfine constant. When the
kaon is bound to a soliton, there is a "transmutation" of quantum numbers: the isospin of the kaon is transmuted to a spin without change of statistics. This
remarkable phenomenon is responsible for the correct assignment of the hyperon
quantum numbers.

For instance if the kaon is bound in the $\Lambda = \frac{1}{2}$ channel, the isospin is zero and the
spin $\frac{1}{2}$, it is like an s-quark except that it carries no baryon charge and hence
behaves bosonic.
It follows then that the $\mathcal{O}(1/N_c)$ contribution to the total energy of the system is

$$\left( \mathbf{J}_{\text{sol}} + c \, \mathbf{J}_K \right)^2 / 2 \tilde{\Omega}$$

where $\mathbf{J}_{\text{sol}}$ is the (collective) angular momentum of the rotating soliton.

The spin of the kaon field can be expressed in terms of its total strangeness as *the bound kaons are to be quantized as bosons* though they carry a half-integer spin. This is obviously a consequence of the fact that the kaon carries no baryonic (or fermionic) charge. Thus, the spin of the kaon is

$$J_K = |S| / 2$$

$S$ spin-$\frac{1}{2}$ bosons in a total symmetric wavefunction

If one ignores interactions between the kaons when there are more than one, then the total energy of the spinning kaon-soliton system (for $S=0,-1,-2,-3$ states) can be expressed as

$$M_{T,J,S} = M_{\text{sol}} + \epsilon |S| + \frac{1}{2 \tilde{\Omega}} \left[ c J(J+1) + (1-c)I(I+1) + \frac{c(c-1)}{4} |S|(|S|+2) \right]$$

where $\mathbf{J} = \mathbf{J}_{\text{sol}} + \mathbf{J}_K$ and $\mathbf{I} = \mathbf{J}_{\text{sol}}$ correspond to the total spin and isospin.
There is only one other bound state. It is in the $(\Lambda, l) = (1/2, 0)$. It describes the $\Lambda(1405)$. (Schat,NNS,Gobbi Nucl. Phys. A585 (1995) 627).

For reasonable model parameters there is no $S=+1$ bound state (or resonance), i.e. there appears to be no pentaquark in this scheme. (NNS, Phys.Lett.B236(1990)245; Klebanov et al, Nucl. Phys. B684 (2004) 264)
Extension to heavier flavors

The bound state picture can be extended to describe charm or bottom baryons, provided strong flavor breaking effects are taken into account (heavy quark symmetry). For example, typical results in the charm sector are

QM: Quark model
Capstick-Isgur ’86

NSM: Naïve BSA
calculation
(no HQS)

SM: BSA with HQS

Scoccola,
arXiv:0905.2722
(review article)
Minimum energy configurations with $B > 1$ have been also studied. In general the corresponding configurations have discrete symmetries.

Some of these configurations have been quantized in order to compare with actual nuclei. For example for $B=4$ (Manko, Manton, Wood, Phys.Rev.C76, 055203)

For $B > 10$ the effect of the finite pion mass seems to be relevant. For example, for $B=12$ different lowest energy configurations were found (Battye and Sutcliffe, Phys. Rev.C73(2006)055205)

The quantization of these configurations has been also undertaken (Battye, Manton, Sutcliffe and Wood, arXiv:0905.0099)
Large $N_c$ QCD gives support to the idea that baryons can be understood as solitonic excitations of a non-linear meson theory (Skyrme-like models).

In its simplest version the model has only two parameters and gives predictions whose precision is within 30%.

The Skyrme model can be generalized by including other mesons in the effective lagrangian (vector mesons, etc).

The model can be generalized to 3 or more flavors. In this case one has to include terms that break flavor symmetry. Those terms should be properly treated. For heavy flavors, heavy quark symmetry has to be taken into account.

It is possible to study soliton configurations with $B > 1$ (multiskyrmiones). In general the lowest energy solutions (at least for $B < 22$) have discrete symmetries.