Finite temperature and density in non-local chiral quark models

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Abstract. Chiral quark models with non-local covariant separable interactions at finite temperature and chemical potential are investigated. We develop a formalism in which the different quark properties are evaluated taking into account the analytic structure of the quark propagator. In this framework we study the chiral restoration phase transition for some definite non-local regulators. We find that in all cases the chiral transition is of first order for low values of $T$, turning into a smooth crossover at a certain “end point”. Using model parameters which lead to the physical pion mass and decay constant, we find for the position of this “end point” the values $(T_E, \mu_E) \approx (60-70, 180-210) \text{MeV}$.

INTRODUCTION

The understanding of the behaviour of strongly interacting matter at finite temperature and/or density is of fundamental interest and has important applications in cosmology, in the astrophysics of neutron stars and in the physics of relativistic heavy ion collisions. From the theory of the quark-gluon interactions, Quantum Chromo Dynamics (QCD), we believe that at zero temperature and density chiral symmetry is spontaneously broken and quarks are confined within hadrons. However, since QCD is asymptotically free, when either the temperature $T$ or the chemical potential $\mu$ are high the effective coupling becomes small. Thus, one expects that at a certain point the system undergoes a phase transition (or crossover) to a new phase in which color is screened rather than confined and chiral symmetry is restored. Unfortunately, so far, it has not been possible to obtain detailed information about the corresponding $T - \mu$ phase diagram directly from QCD. In fact, lattice simulations, which work well for zero density and finite temperature, have serious difficulties in dealing with the complex fermion determinant that arises at finite chemical potential[1]. In this situation, different models have been used to study this sort of problems. Among them the Nambu-Jona-Lasinio model[2] is one of the most popular. In this model the quark fields interact via local four point vertices which are subject to chiral symmetry. If such interaction is strong enough the chiral symmetry is spontaneously broken and pseudoscalar Goldstone bosons appear. It has been shown by many authors that when the temperature and/or density increase, the chiral symmetry is restored[3]. Some covariant nonlocal extensions of the NJL model have been studied in the last few years[4]. Nonlocality arises naturally in the context of several of the most successful approaches to low-energy quark dynamics as, for example, the instanton...
liquid model[5] and the Schwinger-Dyson resummation techniques[6]. It has been also argued that nonlocal covariant extensions of the NJL model have several advantages over the local scheme. Namely, the effective interaction is finite to all orders in the loop expansion and therefore there is not need to introduce extra cut-offs, soft regulators such as Gaussian functions lead to small NLO corrections[7], etc. In addition, it has been shown[8] that a proper choice of the nonlocal regulator and the model parameters can lead to some form of quark confinement, in the sense of a quark propagator without poles at real energies. In this contribution we discuss the extension of this type of models to finite temperature and chemical potential.

FORMULATION

Our starting point is the partition function at zero $T$ and $\mu$,

$$Z_0 = \int D\bar{\psi} D\psi \; e^{-S_E},$$

(1)

where $S_E$ stands for the Euclidean action

$$S_E = \int d^4x \left[ \bar{\psi}(x) (-i \slashed{\partial} + m_c) \psi(x) - \frac{G}{2} j_a(x) j_a(x) \right].$$

(2)

Here $m_c$ is the current quark mass, and the euclidean operator $\slashed{\partial}$ is defined as

$$\slashed{\partial} = \gamma_4 \frac{\partial}{\partial \tau} + \gamma_i \cdot \vec{\nabla}$$

(3)

with $\gamma_4 = i\gamma_0$, $\tau = it$. The current $j_a(x)$ is given by

$$j_a(x) = \int d^4y \; d^4z \; r(y-x) \; r(x-z) \; \bar{\psi}(y) \Gamma_a \psi(z),$$

(4)

where $\Gamma_a = (1, i\gamma^5 \vec{\tau})$ and $r(x-y)$ is a non-local regulator function. The regulator is local in momentum space, namely

$$r(x-z) = \int \frac{d^4p}{(2\pi)^4} \; e^{-i(x-z) \cdot p} \; r(p).$$

(5)

In fact, Lorentz invariance implies that $r(p)$ can only be a function of $p^2$, hence we will use for the Fourier transform of the regulator the form $r(p^2)$ from now on.

To proceed it is convenient to deal with bosonic degrees of freedom. Let us perform a standard bosonization of the theory, introducing the sigma and pion meson fields

$$M_a(x) = (\sigma(x), \vec{\pi}(x)).$$

(6)

Following the usual steps we obtain a partition function equivalent to that in (1):

$$Z_0 = \int D\sigma D\pi \; \det A(M_a) \; \exp \left[ -\frac{1}{2G} \int \frac{d^4p}{(2\pi)^3} \; M_a^2(p) \right],$$

(7)
where the operator $A$ in momentum space reads

$$A(M_a) = \left( -\not{p} + m_c \right) (2\pi)^4 \delta^{(4)}(p - p') + r(p^2) M_a(p - p') r(p'^2) \Gamma_a. \quad (8)$$

At this stage, we perform the mean field approximation by expanding the meson fields around their translational invariant vacuum expectation values $\tilde{\sigma}$ and $\tilde{\pi}_i$ and neglecting the fluctuations $\delta \sigma(x)$ and $\delta \pi_i(x)$. The mean values of the pion fields $\tilde{\pi}_i$ vanish due to symmetry reasons. Within this approximation the determinant in (7) is formally given by

$$\det A = \exp \text{Tr} \log A = \exp V(4) \int \frac{d^4 p}{(2\pi)^4} \text{tr} \log \left[ -\not{p} + m_c + \tilde{\sigma} r^2(p^2) \right], \quad (9)$$

where $\text{tr}$ stands for the trace over the Dirac, flavor and color indices, and $V(4)$ is the four-dimensional volume of the functional integral.

Now the corresponding partition function in the grand canonical ensemble for finite temperature $T$ and chemical potential $\mu$ can be obtained through the replacement in the integrals in (7) and (9)

$$\int \frac{d^4 p}{(2\pi)^4} F(p_A, \vec{p}) \rightarrow \sum_p F(p_A, \vec{p}) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} F(\omega_n - i\mu, \vec{p}) \quad (10)$$

where $\omega_n$ are the Matsubara frequencies corresponding to fermionic modes, $\omega_n = (2n + 1) \pi T$. In performing this replacement we have assumed that the regulator acquires an explicit $\mu$ dependence, as it is the case e.g. in the instanton liquid model. In the same way we replace the volume $V(4)$ by $V/T$, $V$ being the three-dimensional volume in coordinate space. Finally, the grand canonical thermodynamic potential in the mean field approximation is given by

$$\Omega_{MF}(T, \mu) = -\frac{T}{V} \log Z_{MF}(T, \mu) = -4N_c \left( \sum_p \log \left[ p^2 + \Sigma^2(p^2) \right] \right) + \frac{\sigma^2}{2G} \quad (11)$$

with the quark selfenergy $\Sigma(p^2)$ defined as

$$\Sigma(p^2) = m_c + \tilde{\sigma} r^2(p^2). \quad (12)$$

It can be shown that in general this quantity turns out to be divergent. However, it can be regularized by subtracting the corresponding value at zero $T$ and $\mu$, $\Omega^{(0)}_{MF} = \Omega_{MF} - \Omega^{(0)}_{MF}$. Now the minima of the thermodynamic potential are obtained from the solutions of

$$\frac{\partial \Omega_{MF}}{\partial \tilde{\sigma}} = 0 \quad (13)$$

which leads to the following gap equation for $\tilde{\sigma}$:

$$\tilde{\sigma} = 8N_c G \sum_p \frac{\Sigma(p^2) r^2(p^2)}{p^2 + \Sigma^2(p^2)}. \quad (14)$$
Given the thermodynamic potential the expressions for all other relevant quantities can be easily derived. For each flavor the quark vacuum expectation value is given by

\[ \langle \bar{q}q \rangle = \frac{\partial \Omega_{MF}}{\partial m_c} = -4N_c \sum_p \frac{\Sigma(p^2)}{p^2 + \Sigma^2(p^2)}, \]

while the quark density turns out to be

\[ \rho_q = -\frac{\partial \Omega_{MF}}{\partial \mu} = -4iN_c \sum_p \frac{p_4 + \Sigma(p^2) \partial_{p_4} \Sigma(p^2)}{p^2 + \Sigma^2(p^2)}. \]

In obtaining these last two equations it should be noticed that for each quark flavor only half of the kinetic term in Eq. (11) contributes to the derivatives.

As well known[9], if the quark selfenergy is momentum independent as in the conventional NJL model, the sums over the Matsubara frequencies indicated in the previous equations can be easily carried out and one obtains rather simple expressions in terms of the usual occupation numbers. This also holds when the selfenergy only depends on the spatial components of the momentum. However, for the covariant regulators we are considering here the situation is more complicated. The main difficulty is that the analytic structure of the quark propagator \( S(q) = \frac{1}{(-\bar{q} + \Sigma(q^2))} \) in the complex plane can be much richer in this case: there might be a rather complicate structure of poles and cuts.

In Minkowski space and for \( \bar{q} = 0 \) the positions of the poles are given by the solutions of

\[ q_0^2 - [m_c + \bar{\sigma} r^2(-q_0^2)]^2 = 0. \]

Let us denote the real and imaginary parts of these solutions with \( r_p \) and \( i_p \) respectively. For nonzero spatial momentum \( \bar{q} \) it is easy to see that the poles are located at \( R_p + iI_p \), where

\[ R_p = \pm \varepsilon_p, \quad I_p = \pm \frac{r_p i_p}{\varepsilon_p}, \]

with

\[ \varepsilon_p = \sqrt{\frac{r_p^2 - i_p^2 + \bar{q}^2 + \sqrt{(r_p^2 - i_p^2 + \bar{q}^2)^2 + 4r_p^2 i_p^2}}{2}}. \]

On the other hand, the possible cuts of the propagator will be given in general by the cuts of the regulator as a function of \( q \). As in the case of the poles, it is convenient to determine first their position in the \( q_0 \) plane for \( \bar{q} = 0 \). Then it is simple to find the position in the general case applying the above equations to all the points along the cuts.

Before going into the explicit evaluation of the Matsubara sums it is useful to analyze the pole and cut configurations in some relevant situations. One important case is that in which the regulator is such that the Minkowski quark propagator has an arbitrary set of poles but no cuts. Within this class of regulators it might exist a situation in which there are no poles along the real axis. As already mentioned, this situation might be interpreted as a realization of confinement[8]. In that case only quartets of poles located at \( \alpha_p = r_p \pm i i_p, \alpha_p = -r_p \pm i i_p \) appear. On the other hand, if purely real poles do exist...
they will show up as doublets $\alpha_p = \pm r_p$. It is clear that the number and position of the poles depend on the details of the regulator. For example, if we assume it to be a step function as in the standard NJL model, only two purely real poles at $\pm M$ appear, with $M$ being the dynamical quark mass. For a Gaussian interaction, i.e. a regulator of the form

$$r(q^2) = \exp\left(-q^2/2\Lambda^2\right),$$

three different situations might occur. These can be classified according to the value of $\Sigma(0)$ at zero $T$ and $\mu$, which we denote by $\tilde{\Sigma}(0)$. For values of $\tilde{\Sigma}(0)$ below a certain critical value $\tilde{\Sigma}(0)\text{crit}$, two pairs of purely real simple poles and an infinite set of quartets of complex simple poles appear. At $\tilde{\Sigma}(0) = \Sigma(0)\text{crit}$, the two pairs of purely imaginary simple poles turn into a doublet of double poles with $i_p = 0$, while for $\tilde{\Sigma}(0) > \tilde{\Sigma}(0)\text{crit}$ only an infinite set of quartets of complex simple poles is obtained. There are some other physically interesting regulators as the Lorentzian regulators

$$r(q^2) = \frac{1}{1 + (q^2/\Lambda^2)^n},$$

which belong to the “no cuts” group. Unfortunately some other important ones, like that arising within the instanton liquid model[5], do lead to propagators which present poles and cuts. Thus, it is necessary to consider this more general situation.

The evaluation of the sums over the Matsubara frequencies can be carried out by a convenient choice of integration paths in the complex plane and the help of Cauchy’s theorem. As an example we consider the sum appearing in the gap equation Eq.(14),

$$S = \int \frac{d^3q}{(2\pi)^3} T \sum_{n=-\infty}^{\infty} F[(\omega_n - i\mu)^2 + \bar{q}^2)],$$

where

$$F(q^2) = \frac{\Sigma(q^2)r^2(q^2)}{q^2 + \Sigma^2(q^2)}.$$

Assuming that the regulator is such that the quark propagator at $\bar{q} = 0$ has an arbitrary set of simple poles located at $\pm r_p \pm i i_p$ and a single cut a single cut lying along the real axis in the regions $[-\infty, -r_c]$ and $[r_c, \infty]$ one gets[10]

$$S = \int \frac{d^4q}{(2\pi)^4} F(q^2) + 2 \int \frac{d^3q}{(2\pi)^3} \text{Re} \sum_{\text{poles}} \gamma_p \text{Res}[F(q^2 - z^2); z_p] \left[n_+(z_p) + n_-(z_p)\right]$$

$$- \frac{1}{\pi} \int \frac{d^3q}{(2\pi)^3} \int_{\sqrt{r_c^2 + q^2}}^{\infty} dx \text{Im}\left\{F(q^2 - x^2 - i\epsilon) \left[n_+(x) + n_-(x)\right]\right\},$$

where the first term corresponds to the value of $S$ at zero temperature an chemical potential. The coefficient $\gamma_p$ is defined as $\gamma_p = 1/2$ for $\text{Im}(z_p) = 0$ and $\gamma_p = 1$ otherwise and the residues are evaluated at $z_p = \pm r_p \pm i i_p / e_p$. The sum runs over the poles with $\text{Re}(z_p) > 0$ and $\text{Im}(z_p) \geq 0$. Finally, $n_{\pm}(z)$ stand for the occupation number functions

$$n_{\pm}(z) = \frac{1}{1 + \exp[(z \mp \mu)/T]}.$$
It is clear that the steps leading to Eq.(24) can be also followed to evaluate the sums appearing in e.g. the quark vacuum expectation value and the quark density, Eqs.(15) and (16) respectively, once the function $F(q^2)$ is properly redefined. In the case of the regularized thermodynamic potential $\Omega_{Mf}^{(r)}$, the situation is somewhat more complicated, since the argument of the sum includes a logarithm that introduces cuts in the $\zeta$ plane outside the real axis. Nevertheless a similar relation can be derived to calculate the corresponding Matsubara sum.

RESULTS

Having introduced the formalism needed to deal with models with non-local separable interactions at finite temperature and chemical potential, we turn now to the results of numerical calculations carried out for some particular regulators. Here we present in detail those corresponding to the Gaussian regulator. Results for the Lorentzian and the instanton model regulators are qualitatively similar and can be found in Ref.[10]. We consider two sets of values for the parameters of the model. Set I corresponds to $G = 50 \text{ GeV}^{-2}$, $m_c = 10.5 \text{ MeV}$ and $\Lambda = 627 \text{ MeV}$, while for Set II the respective values are $G = 30 \text{ GeV}^{-2}$, $m_c = 7.7 \text{ MeV}$ and $\Lambda = 760 \text{ MeV}$. Both sets of parameters lead to the physical values of the pion mass and decay constant. For Set I the calculated value of the chiral quark condensate at zero temperature and chemical potential is $-200 \text{ MeV}^3$, while for Set II it is $-220 \text{ MeV}^3$. These values are similar in size to those determined from lattice gauge theory or QCD sum rules. The corresponding results for the self-energy at zero momentum are $\Sigma(0) = 350 \text{ MeV}$ for Set I and $\Sigma(0) = 300 \text{ MeV}$ for Set II. It is possible to check that Set I corresponds to a situation in which the quark propagator has no purely real poles and Set II to the case in which there are two pairs of them. Following Ref.[8], Set I might be interpreted as a confining one since quarks cannot materialize on-shell in Minkowski space.

The behaviour of the zero-momentum self-energy $\Sigma(0)$, the chiral condensate and the quark density as function of the chemical potential for some values of the temperature is shown in Fig. 1. There, we observe that at $T = 0$ there is a first order phase transition for both the confining and the non-confining sets of parameters. As the temperature increases, the value of the chemical potential at which the transition shows up decreases. Finally, above a certain value of the temperature the first order phase transition does not longer exist and, instead, there is a smooth crossover. This phenomenon is clearly shown in the left panels of Fig. 2, where we display the critical temperature at which the phase transition occurs as a function of the chemical potential (upper left panel) and as a function of the density (lower left panel). The point at which the first order phase transition ceases to exist is usually called “end point”. In the chiral limit the latter turns into the so-called “tricritical point”, which is the point at which the second order phase transition expected to happen in QCD with two massless quarks becomes a first order one. In fact, this is also what happens within the present model in the chiral limit, as it is shown in the right panels of Fig. 2. Some predictions about both the position of this point and its possible experimental signatures exist in the literature[11]. In our case the “tricritical point” is located at $(T_p, \mu_p) = (70 \text{ MeV}, 130 \text{ MeV})$ for Set I and $(70 \text{ MeV}, 140 \text{ MeV})$
FIGURE 1. Behaviour of the self-energy, the chiral condensate and the quark density as functions of the chemical potential for three representative values of the temperature. Full line corresponds to $T=0$, dashed line to $T = 50$ MeV and dotted line to $T = 100$ MeV. The left panels display the results for Set I and the right panels those for Set II. The quark density $\rho$ is given with respect to nuclear matter density $\rho_0 = 1.3 \times 10^6$ MeV$^3$.

for Set II, while the “end points” are placed at $(T_E, \mu_E) = (70$ MeV, $180$ MeV) and $(55$ MeV, $210$ MeV), respectively.

It is interesting to discuss in detail the situation concerning the confining set. In this case we can find, for each temperature, the chemical potential $\mu_d$ at which confinement is lost. Following the proposal of Ref.[8], this corresponds to the point at which the
FIGURE 2. Critical temperatures as a function of the chemical potential (upper panels) and as a function of the quark density (lower panels). The right panels corresponds to the chiral limit and the left panels to the case of finite quark masses. In both panels the full line stands for a first order phase transition while in the right panel the dotted line indicates that the transition is of second order. The dashed line in the upper right panel indicates the temperature at which, for Set I, complex poles of the propagator turn into real poles.

Self-energy at zero momentum reaches the value $\Sigma(0)_{\text{crit}}$. Using the values of $m_c$ and $\Lambda$ corresponding to Set I we get $\Sigma(0)_{\text{crit}} = 267$ MeV. For low temperatures, $\mu_d$ coincides with the chemical potential at which the chiral phase transition takes place. However, for a temperature close enough to that of the “end point”, $\mu_d$ starts to be slightly smaller than the value of $\mu$ that corresponds to the chiral restoration. Above $T_E$, it is difficult to make an accurate comparison since, for finite quark masses, the chiral restoration proceeds through a smooth crossover. However, we can still study the situation in the chiral limit. In this case we find that, in the region where the chiral transition is of second order, deconfinement always occurs, for fixed $T$, at a lower value of $\mu$ than the chiral transition. The corresponding critical line is indicated by a dashed line in the upper right panel of Fig. 2. In any case, as we can see in this figure, the departure of the line of chiral restoration from that of deconfinement is in general not too large. This indicates that within the present model both transitions tend to happen at, approximately, the same point.
CONCLUSIONS

In this work we investigate the features of the chiral phase transition in some non-local chiral quark models with covariant separable interactions. We consider several types of regulators: the Gaussian regulator, the Lorentzian regulator and the instanton liquid model regulator. We find that in all the cases the phase diagram is quite similar. In particular, we obtain that for two light flavors the transition is of first order at low values of the temperature and becomes a smooth crossover at a certain “end point”. Our predictions for the position of this point are very similar for both types of regulators and slightly smaller than the values in Refs.[11], $T_P \approx 100$ MeV and $\mu_P \approx 200 – 230$ MeV. In this sense, we remark that our model predicts a critical temperature at $\mu = 0$ of about 100 MeV, somewhat below the values obtained in modern lattice simulations which suggest $T_c \approx 140 – 190$ MeV. In any case, our calculation seems to indicate that $\mu_P$ might be smaller than previously expected even in the absence of strangeness degrees of freedom.

Several extensions of this work are of great interest. For example, it would be very important to investigate the impact of the introduction of strangeness degrees of freedom and flavor mixing on the main features of the chiral phase transition. Also, the competition between chiral symmetry breaking and color superconductivity at large chemical potential deserves further studies. Work along these lines is under way.

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REFERENCES

1. Karsch, F. hep-lat/0106019.