Cranking in isospace
– introducing a new concept of the symmetry energy

Introduction to iso cranking model
Investigation of the symmetry energy in Skyrme HF and RMF
Origin of the linear term in mean field models
New insight into the symmetry energy

\[
\frac{E}{A} = -aV + \frac{a_S}{A^{1/3}} + \left[ a_{sym}^{(V)} - \frac{a_{sym}^{(S)}}{A^{1/3}} + \ldots \right] \left( I^2 + \lambda \frac{I}{A} \right) + \ldots,
\]

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Symmetry Energy

\[ E_{\text{sym}} = \frac{1}{2} a_{\text{sym}} T^2 = \frac{1}{2} (a_{\text{kin}} + a_{\text{int}}) T^2. \]

Groundstate in nuclei have lowest \(T\), \(\langle T \rangle = T_z = \frac{1}{2} (N-Z)\)

Bethe-Weizsäcker mass formula: \(\sim (N-Z)^2\)

\[
\begin{align*}
E_{\text{sym}} &= \frac{1}{2} a_{\text{sym}} T(T+1) \\
a_{\text{sym}} &= \frac{1}{2} a_{\text{vol}} / A - \frac{1}{2} a_{\text{surf}} / A^{4/3} \\
&= 134.4 / A - 203.6 / A^{4/3}
\end{align*}
\]

**Iso Spin Cranking Model**

**Single-particle Routhian:**
\[ \hat{H}^\omega = \hat{H}_{sp} - \omega_T \hat{T} \]

**Single-particle Hamiltonian:**
Equidistant level, fourfold degenerate (isospin and Kramers)
\[ E_{sp} = \sum_i e_i = \sum_i (i\varepsilon) \]

**Iso-cranking term:**
removes the isospin degeneracy.
\[ -\omega_T \hat{T} \]

**Energy**
\[ E \equiv E^\omega + \hbar \omega_T T_x \]
\[ E = \frac{1}{2} \delta e T_x^2 \]

\[ |\pm\rangle = \frac{1}{\sqrt{2}} (|n\rangle \pm |p\rangle) \]

Iso Spin Cranking Model

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Symmetry energy in the mean field

\[ E_{\text{sym}} = \frac{1}{2} a_{\text{sym}} T^2 = \frac{1}{2} (a_{\text{kin}} + a_{\text{int}}) T^2. \]

- Any bi-fermionic system is characterised by a symmetry energy, coming from the discretness of the s.p. levels (no assumption of any force!) This term is proportional to the average level spacing:

\[ E = \frac{1}{2} \varepsilon T^2 \quad \varepsilon \approx 2 \frac{\pi^2}{3A} \approx 16 \frac{\pi^2}{3A} - 20 \frac{\pi^2}{3A} \approx \frac{53}{A} \frac{66}{A} \text{ MeV}. \]

- The nuclear interaction differs between states of different iso-spin – resulting in an additional iso vector potential. This potential can be obtained e.g. from an interaction

\[ V_{TT} = \frac{1}{2} \kappa \hat{T} \cdot \hat{T}. \]

- This interaction leads to a term \( E = \kappa T^2 \), i.e. \( \kappa T^z^2 = \kappa \frac{1}{4} (N-Z)^2 \) (Hartree approx) Taking into account the exchange term (Fock), \( E = \kappa T(T+1) \) (see e.g B&M, vol 1)

\[ E_{\text{sym}} = \frac{1}{2} (\varepsilon + \kappa) T^2 + \frac{1}{2} \kappa T \]
Invesigate this concept in Skyrme HF-BCS

- The Skyrme HF can be divided in an iso scalar $\Gamma_0$ and iso vector potential $\Gamma_1$. There are 5 isoscalar and 5 isovector densities and related coupling constants.

$$\sum_{t=0,1} \int d^3r \, \mathcal{H}_t(r):$$

$$\mathcal{H}_t(r) = C^\rho_t \rho_t^2 + C^{\Delta \rho}_t \rho_t \Delta \rho_t + C^\tau_t \rho_t \tau_t$$

$$+ C^J_t \mathbf{J}_t^2 + C^{\nabla J}_t \rho_i \nabla \cdot \mathbf{J}_t$$

- Switch off the iso vector part and calculate the average spacing as a function of $(N-Z)$
- Determine $\kappa$ via calculating the full functional
Investigate this concept in Skyrme HF-BCS

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$$\sum_{t=0,1} \int d^3r \, \mathcal{H}_t(r):$$

$$\mathcal{H}_t(r) = C_i^0 \rho_i^2 + C_i^{A\rho} \rho_t \Delta \rho_t + C_i^\tau \rho_t \tau_t$$

$$+ C_i^J J_t^2 + C_i^{\nabla J} \rho_i \nabla \cdot J_t$$

- Switch off the isovector part and calculate the average spacing as a function of $(N-Z)$
- Determine $\kappa$ via calculating the full functional
- How does the linear term behave
Symmetry energy in Skyrme HF

\[ \Delta \tilde{E}_T^{(HF)} = \tilde{E}_T^{(HF)} - \tilde{E}_{T=0}^{(HF)} \approx \frac{1}{2} \varepsilon T^2. \]

The different Skyrme forces have different effective mass. Once the level spacing \( \varepsilon \) is corrected for the effective mass, \( \varepsilon' = \frac{m^*}{m} \varepsilon \) the coefficients become very similar. Shaded area corresponds to average spacing

No linear term!
Spread in kinetic energy
Skyrme iso vector potential

- The Skyrme functional has an isovector potential that is proportional to $T(T+1)$ and can be characterised by a single coefficient, $\kappa$.

- The smaller the effective mass, the smaller the iso vector potential!
Global fit to $\varepsilon$ and $\kappa$

$$\varepsilon(A) = \frac{\varepsilon_V}{A} - \frac{\varepsilon_S}{A^{4/3}};$$

$$\kappa(A) = \frac{\kappa_V}{A} - \frac{\kappa_S}{A^{4/3}};$$
why is $\varepsilon^* = \kappa^*$?
Symmetry energy in SHF

- Symmetry energy obtained as

\[ E_{\text{sym}}^{(\text{SHF})} = \frac{1}{2} \varepsilon(A, T_z) T^2 + \frac{1}{2} \kappa(A, T_z) T(T + 1), \]

- Value of \( a_v (a_{\text{sym}} = a_v/A + a_s/A^{4/3}) \) close to value from infinite nuclear matter

- Fundamental property that \( \varepsilon^* = K^* \) ?
Test the same concept in RMF

\[ V_{tot} = V(\mathbf{r}) + \beta S(\mathbf{r}) = g_\omega \omega^0(\mathbf{r}) + g_\rho \vec{\rho} \cdot \vec{\rho}^0(\mathbf{r}) + \beta g_\sigma \sigma(\mathbf{r}). \]

\[ V_{is}(\mathbf{r}) = g_\omega \omega^0(\mathbf{r}) + \beta g_\sigma \sigma(\mathbf{r}), \]
\[ V_{iv}(\mathbf{r}) = g_\rho \vec{\rho} \cdot \vec{\rho}^0(\mathbf{r}). \]

\[ \tilde{E}_T(A, T_z) - \tilde{E}_{T=0}(A, T_z = 0) = \frac{1}{2} \varepsilon(A, T_z) T^2. \]

- Strong interaction – disregard Coulomb
- What is the size of the linear term?
- What are the values of \( a_{\text{sym}} = a_V/A + a_S/A^{4/3} \)
Formalism to determine $\varepsilon$ and $\kappa$ in RMF

**without isovector meson**

\[ \tilde{E}^{\text{RMF}}_T - \tilde{E}^{\text{RMF}}_{T=0} = \frac{1}{2} \varepsilon T^2 \]

**with isovector meson**

\[ E^{\text{RMF}}_T - \tilde{E}^{\text{RMF}}_T = \frac{1}{2} \kappa T^2 \text{ or } \frac{1}{2} \kappa T(T + 1) \]

**Effect of pairing**

\[ E^{\text{RMF} + BCS}_T - \tilde{E}^{\text{RMF} + BCS}_T = \frac{1}{2} \kappa T^2 \text{ or } \frac{1}{2} \kappa T(T + 1) \]
Symmetry energy in RMF from level spacing

- $\epsilon$ is constant for large $T_z$

- After effective mass scaling $m^*/m$ within imperical limits
Determine $\kappa$ from the full Lagrangian, including the $\rho$-meson.
\[ E_{\text{sym}} \approx \frac{1}{2} \varepsilon T^2 + \frac{1}{2} \kappa T (T + 1 + \varepsilon / \kappa) \approx \frac{1}{2} (\varepsilon + \kappa) T (T + 1). \]

The nuclear symmetry energy in RMF, \( E_{\text{sym}} = a T(T+1) \)
The nuclear symmetry energy in RMF follows rather closely the values by Duflo Zuker.

\[ a_{\text{sym}}^{(\text{RMF})} = \frac{133.20}{A} - \frac{220.27}{A^{4/3}} \text{ [MeV]}, \]

\[ a_{\text{sym}}(A) = \frac{134.4}{A} - \frac{203.6}{A^{4/3}} \text{ [MeV]}. \]
Symmetry energy in RMF

- Concept of the symmetry energy composed by two terms, the average level spacing at the Fermi surface
- The volume term of the symmetry energy $a_v (a_{\text{sym}} = a_v/A + a_s/A^{4/3})$ determined in finite nuclei is much smaller than the one in infinite nuclear matter $a_v (a_{\text{sym}} = a_v/A + a_s/A^{4/3})$
- Surprisingly, the RMF theory which is a Hartree approximation generates a symmetry energy $E_{\text{sym}}$ that is fitted nicely by a $T(T+1)$ dependence.
2. Formalism

Lagrangian density in RMF theory

\[ \mathcal{L} = \bar{\psi} \left[ i \gamma^\mu \partial_\mu - m - g_\sigma \sigma - g_\omega \gamma^\mu \omega_\mu - \frac{1}{2} g_\rho \gamma^\mu \tau_B \cdot \rho_\mu - e \gamma^\mu \frac{1 - \tau_3}{2} A_\mu \right] \psi \]

\[ + \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - U(\sigma) \]

\[ - \frac{1}{4} \omega^\mu \omega_\mu + \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu + U(\omega) \]

\[ - \frac{1}{4} \rho^\mu \cdot \rho_\mu + \frac{1}{2} m_\rho^2 \rho^\mu \cdot \rho_\mu + U(\rho) \]

\[ - \frac{1}{4} A^\mu \rho A_\mu \]

\[ U(\sigma) = \frac{1}{3} g_2 \sigma^3 + \frac{1}{4} g_3 \sigma^4, \]

\[ U(\omega) = \frac{1}{4} c_3 (\omega^\mu \omega_\mu)^2, \]

\[ U(\rho) = \frac{1}{4} \delta_3 (\rho^\mu \cdot \rho_\mu). \]

\[ \begin{cases} 
\omega_{\mu\nu} \equiv \partial_\mu \omega_\nu - \partial_\nu \omega_\mu, \\
\rho_{\mu\nu} \equiv \partial_\mu \rho_\nu - \partial_\nu \rho_\mu - 2 g_\rho \rho^\mu \times \rho^\nu, \\
A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. 
\end{cases} \]

\( \sigma \)-isoscalar scalar meson, offering attractive interaction between the nucleon and nucleon

\( \omega \)-isoscalar vector meson, offering repulsive interaction between the nucleon and nucleon

\( \rho \)-isovector vector meson, offering isospin symmetry

A isospin-related photon, offering Coulomb interaction, isovector
Structure of Nucleonic Pairs

- $N=Z \implies$ (almost) identical wavefunctions
- particle particle interaction between pairs with identical orbits
- Pauli Principle

Isovector Pairs $T=1, S=0$

Isoscalar Pairs $T=0, S=1$
Unitary BCS Transformation

From particles to quasiparticles

\[ a_i^\dagger = \begin{pmatrix} a_i^{\dagger p} \\ a_i^{\dagger n} \\ a_i^{\dagger p} \\ a_i^{\dagger n} \end{pmatrix} \quad \begin{pmatrix} \alpha_i^\dagger \\ \alpha_i \end{pmatrix} = \begin{pmatrix} U_i & -V_i \\ -V_i^* & U_i^* \end{pmatrix} \begin{pmatrix} a_i^\dagger \\ a_i \end{pmatrix} \quad \alpha_i^\dagger = \begin{pmatrix} \alpha_i^{\dagger 1} \\ \alpha_i^{\dagger 2} \\ \alpha_i^{\dagger 3} \\ \alpha_i^{\dagger 4} \end{pmatrix} \]

Mixing creation and annihilation operators for n and p
and choose the BCS variational parameters

\[ U_i = u_i I_4 \quad V_i = \begin{pmatrix} 0 & v_{i1} & v_{i2} & v_{i3} \\ -v_{i1} & 0 & v_{i3}^* & -v_{i2} \\ -v_{id} & -v_{i3}^* & 0 & v_{i1} \\ -v_{i3} & v_{i2} & v_{i1} & 0 \end{pmatrix} \]

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Nucl. Phys. A186 (1972) 475

Density matrix \( \rho \) and pairing tensor \( \kappa \)

\[ b^s = \mathbf{\Lambda}_s \mathbf{\Lambda}^s = \mathbf{\Omega}_s^s \mathbf{\Gamma}^s \quad \overline{v_i}^2 = v_{i1}^2 + v_{i2}^2 + |v_{i3}|^2 \]

To go from BCS→HFB, replace \( U_i \rightarrow U \) and \( V_i \rightarrow V \) by matrices with dimension 4 \( N \)
Investigate the generalised pairing hamiltonian

Employ approximate number projection via L.N.

\[ \hat{H}_\omega = \hat{H}_\omega - \sum_\tau \lambda^{(1)}_\tau \Delta \hat{N}_\tau - \sum_{\tau\tau'} \lambda^{(2)}_{\tau\tau'} \Delta \hat{N}_\tau \Delta \hat{N}_{\tau'} \]

Investigate the BCS- and HFB solution as a function of strength

- BCS $G^{T=0}/G^{T=1} =$? and HFB $G^{T=0}/G^{T=1} =$?
Energy (Routhian)

\[ \mathcal{E} = \langle \sum_{i>0} \epsilon_i (a_{i \uparrow}^\dagger a_{i \downarrow} + a_{i \uparrow}^\dagger a_{i \downarrow}^\dagger + a_{i \downarrow}^\dagger a_{i \uparrow} + a_{i \downarrow}^\dagger a_{i \uparrow}^\dagger) + H_{BCS} (+\omega J_x) \rangle \]

Solve the variational equation

(Constrain proton and neutron number \( \rightarrow \) gives Fermi level)

\[ \delta(\mathcal{E} - \lambda \langle \hat{N} + \hat{Z} \rangle) = 0 \]

Occupation numbers; quasi-particle energies

\[ \overline{v}_i^2 = \frac{1}{2} \left( 1 - \frac{\epsilon_i - \lambda}{E_i} \right) \quad E_i = \sqrt{(\epsilon_i - \lambda)^2 + \Delta^2} \]

Pairing Gaps:

\[ \Delta_{1 \pm 1} = G^{t=1} \sum_{i>0} u_i \overline{v}_i \]

\[ \Delta_{10} = G^{t=1} \sum_{i>0} u_i \overline{R} v_i \]

Usual Gap, \( n-\tilde{n}, p-\tilde{p} \)

\[ \Delta^2 = \Delta_{1 \pm 1}^2 + \Delta_{10}^2 + \Delta_0^2 + \tilde{\Delta}_0^2 \]

\[ \Delta_0 = G^{t=0} \sum_{i>0} u_i \overline{s} v_i \]

\[ \tilde{\Delta}_0 = \tilde{G}^{t=0} \sum_{i>0} u_i v_i \]

\[ \tilde{G}^{t=0} = 0 \rightarrow v_{i1} = 0 \]

Gap \( T=0 \) \( a\tilde{a} \)

Gap \( T=0 \) \( aa \)

T=1 Gap, \( \tilde{n}-p + p-\tilde{n} \)
The alignment in isospace $tx$ and the response of $T=1$ and $T=0$ pair field. Calculations for $24$Mg and $48$Cr.

*Meissner effect in isospace!*
Competition between 2qp excitation and symmetry energy in o-o nuclei

T=0 states in o-o nuclei are 2qp excitations $\sim 1/\sqrt{A}$

T=1 states have larger symmetry energy $\sim 1/A$
Calculations with $T=0$ and $T=1$ pairing. The excitation spectrum changes drastically. $GT=0$ fitted to Wigner energy.
• Define the anisotropy in iso-space as

\[ \Delta = [\Delta_x, \Delta_y, \Delta_z] \]

\[ \Delta_x = \frac{1}{\sqrt{2}} (\Delta_{pp} - \Delta_{nn}) ; \]
\[ \Delta_y = \frac{-i}{\sqrt{2}} (\Delta_{pp} + \Delta_{nn}) ; \]
\[ \Delta_z = \Delta_{pn} . \]

• Depending on the phase angle between proton and neutron pairing, the (non-)collective rotation occurs around the (y)x-axis
How collective is rotation in isospace? Depends on deformation! Apparently, already a small $T=1$ pair gap generates collectivity. Condition that $\Delta > \varepsilon$ well fullfilled in most nuclei.
Calculated excitation spectrum for the T=2 states in e-e nuclei. Only T=1 pairing is present in the calculations. Shell closures are clearly visible!
Iso spin mixing due to T=1 pairing interaction

- T=1 pairing violates iso-spin – resulting in deformation in iso-space
- T=0 pairing restores iso spin (scalar in iso space)
- We need iso spin breaking to calculate iso spin excited states.