NONLOCAL DIFFUSION EQUATIONS

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Non-local diffusion.

Let $J : \mathbb{R}^N \to \mathbb{R}$, nonnegative, smooth with

$$\int_{\mathbb{R}^N} J(r)dr = 1.$$

Assume that is compactly supported and radially symmetric.

Non-local diffusion equation

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t).$$

Non-local diffusion.

In this model, $u(x,t)$ is the density of individuals in x at time t and $J(x-y)$ is the probability distribution of jumping from y to x. Then

$$(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$$

is the rate at which the individuals are arriving to x from other places

$$-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$$

is the rate at which they are leaving from x to other places.

Non-local diffusion.

The non-local equation shares some properties with the classical heat equation

$$u_t = \Delta u.$$

Properties

- Existence, uniqueness and continuous dependence on the initial data.
- Maximum and comparison principles.
- Perturbations propagate with infinite speed. If u is a nonnegative and nontrivial solution, then $u(x,t) > 0$ for every $x \in \mathbb{R}^N$ and every $t > 0$.

Remark.

There is no regularizing effect for the non-local model.

Decay for the heat equation

For the heat equation we have an explicit representation formula for the solution in Fourier variables. In fact, from the equation

$$v_t(x,t) = \Delta v(x,t)$$

we obtain

$$\hat{v}_t(\xi,t) = -|\xi|^2 \hat{v}(\xi,t),$$

and hence the solution is given by,

$$\hat{v}(\xi,t) = e^{-|\xi|^2 t} \hat{u}_0(\xi).$$

From where it can be deduced that

$$\|v(\cdot,t)\|_{L^q(\mathbb{R}^N)} \leq C\, t^{-\frac{N}{2}(1-\frac{1}{q})}.$$

The convolution model

The asymptotic behavior as $t \to \infty$ for the nonlocal model

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t),$$

is given by

Theorem (Chasseigne-Chaves-R.) The solutions verify

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}}.$$

The convolution model

The proof of this fact is based on a explicit representation formula for the solution in Fourier variables. In fact, from the equation

$$u_t(x,t) = (J * u - u)(x,t),$$

we obtain

$$\hat{u}_t(\xi,t) = (\hat{J}(\xi) - 1)\hat{u}(\xi,t),$$

and hence the solution is given by,

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi).$$

From this explicit formula it can be obtained the decay in $L^\infty(\mathbb{R}^d)$ of the solutions. Just observe that

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi) \approx e^{-t}\hat{u}_0(\xi),$$

for ξ large and

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi) \approx e^{-|\xi|^2 t}\hat{u}_0(\xi),$$

for $\xi \approx 0$. Hence, one can obtain

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le Ct^{-\frac{N}{2}}.$$

This decay, together with the conservation of mass, gives the decay of the $L^q(\mathbb{R}^N)$-norms by interpolation. It holds,

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^N)} \leq C\,t^{-\frac{N}{2}(1-\frac{1}{q})}.$$

Note that the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian.

Newmann boundary conditions.

One of the boundary conditions that has been imposed to the heat equation is the *Neumann boundary condition*, $\partial u / \partial \eta (x,t) = 0$, $x \in \partial\Omega$.

Non-local Neumann model

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t) - u(x,t))dy$$

for $x \in \Omega$.

Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω.

Existence, uniqueness and a comparison principle

Theorem (Cortazar - Elgueta - R. - Wolanski)

For every $u_0 \in L^1(\Omega)$ there exists a unique solution u such that $u \in C([0,\infty); L^1(\Omega))$ and $u(x,0) = u_0(x)$.
Moreover the solutions satisfy the following comparison property:

$$\text{if } u_0(x) \le v_0(x) \text{ in } \Omega, \ \text{ then } u(x,t) \le v(x,t) \text{ in } \Omega \times [0,\infty).$$

In addition the total mass in Ω is preserved

$$\int_\Omega u(y,t)\, dy = \int_\Omega u_0(y) dy.$$

Asymptotic behavior

Theorem (Chasseigne - Chaves - R. and Andreu - Mazon - R. - Toledo)

Let $g(x,t) = h(x)$ *such that*

$$0 = \int_\Omega \int_{\mathbf{R}^N \setminus \Omega} G(x-y)h(y)\,dy\,dx.$$

Then there exists a unique solution φ *of the problem*

$$0 = \int_\Omega J(x-y)(\varphi(y)-\varphi(x))\,dy + \int_{\mathbf{R}^N \setminus \Omega} G(x-y)h(y)\,dy$$

that verifies $\int_\Omega u_0 = \int_\Omega \varphi$ *and there exists* $\beta = \beta(J,\Omega) > 0$ *such that*

$$\|u(t)-\varphi\|_{L^2(\Omega)} \le e^{-\beta t}\|u_0-\varphi\|_{L^2(\Omega)}.$$

Asymptotic behavior

If the compatibility conditions does not hold then solutions are unbounded.
Here β_1 is given by

$$\beta_1 = \inf_{u \in L^2(\Omega), \int_\Omega u = 0} \frac{\frac{1}{2} \int_\Omega \int_\Omega J(x-y)(u(y)-u(x))^2 \, dy \, dx}{\int_\Omega (u(x))^2 \, dx}.$$

Approximations

Now, our goal is to show that the Neumann problem for the heat equation, can be approximated by suitable nonlocal Neumann problems.
More precisely, for given J we consider the rescaled kernels

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right),$$

with

$$C_1^{-1} = \frac{1}{2}\int_{B(0,d)} J(z) z_N^2 \, dz,$$

which is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it.

Approximations

Then, we consider the solution $u_\varepsilon(x,t)$ to

$$\begin{cases} (u_\varepsilon)_t(x,t) & = \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega J_\varepsilon(x-y)(u_\varepsilon(y,t)-u_\varepsilon(x,t))\,dy \\ u_\varepsilon(x,0) & = u_0(x). \end{cases}$$

Theorem (Cortazar - Elgueta - R. - Wolanski)

Let $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ be the solution to the heat equation $u_t = \Delta u$ with Neumann boundary conditions $\partial u/\partial\eta(x,t) = 0$ and u_ε be the solution to the nonlocal model. Then,

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^\infty(\Omega)} = 0.$$

Approximations

Idea of why the involved scaling is correct

Let us give an heuristic idea in one space dimension, with $\Omega = (0, 1)$, of why the scaling involved is the right one.

We have

$$u_t(x,t) = \frac{1}{\varepsilon^3}\int_0^1 J\left(\frac{x-y}{\varepsilon}\right)\big(u(y,t)-u(x,t)\big)dy$$
$$:= A_\varepsilon u(x,t).$$

Approximations

If $x \in (0, 1)$ a Taylor expansion gives that for any fixed smooth u and ε small enough, the right hand side $A_\varepsilon u$ becomes

$$A_\varepsilon u(x) = \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{x-y}{\varepsilon}\right)(u(y) - u(x))\, dy$$

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} J(w)\,(u(x - \varepsilon w) - u(x))\, dw$$

Approximations

$$= \frac{u_x(x)}{\varepsilon} \int_{\mathbb{R}} J(w)\, w\, dw + \frac{u_{xx}(x)}{2} \int_{\mathbb{R}} J(w)\, w^2\, dw + O(\varepsilon)$$

As J is even

$$\int_{\mathbb{R}} J(w)\, w\, dw = 0$$

and hence,

$$A_\varepsilon u(x) \approx u_{xx}(x),$$

and we recover the Laplacian for $x \in (0, 1)$.

The $p-$Laplacian

The problem,

$$u_t(t,x) = \int_\Omega J(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy,$$
$$u(x,0) = u_0(x).$$

is the analogous to the p-Laplacian

$$\begin{cases} u_t = \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) & \text{in } (0,T)\times\Omega, \\ |\nabla u|^{p-2}\nabla u\cdot\eta = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Approximations

For given $p \geq 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \quad C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p \, dz.$$

Theorem (Andreu - Mazon - R. - Toledo)

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p \geq 1$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$, $u_0 \in L^p(\Omega)$. Then,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(t,.) - u(t,.)\|_{L^p(\Omega)} = 0.$$

Convective terms

$$\begin{cases} u_t(t,x) = (J * u - u)(t,x) + (G * (f(u)) - f(u))(t,x), \\ u(0,x) = u_0(x) \qquad (\text{ now } x \in \mathbb{R}^N \text{ !!}). \end{cases}$$

Theorem (Ignat - R.)

There exists a unique global solution

$$u \in C([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty); \mathbb{R}^N).$$

Moreover, the following contraction property

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^N)}$$

holds for any $t \ge 0$. In addition, $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}$.

Convective terms

Let us consider the rescaled problems

$$\begin{cases} (u_\varepsilon)_t(t,x) = \dfrac{1}{\varepsilon^{N+2}} \displaystyle\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) (u_\varepsilon(t,y) - u_\varepsilon(t,x))\, dy \\ \qquad + \dfrac{1}{\varepsilon^{N+1}} \displaystyle\int_{\mathbb{R}^N} G\left(\frac{x-y}{\varepsilon}\right) (f(u_\varepsilon(t,y)) - f(u_\varepsilon(t,x)))\, dy, \\ u_\varepsilon(x,0) = u_0(x). \end{cases}$$

Note that the scaling of the diffusion, $1/\varepsilon^{N+2}$, is different from the scaling of the convective term, $1/\varepsilon^{N+1}$.

Convective terms

Theorem (Ignat - R.)

We have

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon - v\|_{L^2(\mathbb{R}^N)} = 0,$$

where $v(t,x)$ is the unique solution to the local convection-diffusion problem

$$v_t(t,x) = \Delta v(t,x) + b \cdot \nabla f(v)(t,x),$$

with initial condition $v(x,0) = u_0(x)$ and $b = (b_1, ..., b_d)$ given by

$$b_j = \int_{\mathbb{R}^N} x_j \, G(x) \, dx, \qquad j = 1, ..., d.$$

Non-local diffusion.

References

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